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Some Multidimensional Spline Approximation Methods

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1. INTRODUCTION

Schoenberg [20] introduced a spline approximation method of the form $Sf(x) = \sum f(\xi_i)\, \phi_i(x)$ where $\phi_i(x)$ are certain spline functions. This method was further studied by Marsden and Schoenberg [16] and Marsden [14, 15], where uniform convergence theorems, order of convergence estimates, variation diminishing properties, and other results analogous to those for Bernstein polynomials are obtained. Marsden [14] and independently Karon [11] and Karlin and Karon [10] generalize these ideas to produce Tchebycheffian spline approximation methods. Scherer [18] exploited the approximation properties of these operators to obtain direct and inverse theorems as well as saturation results for spline approximation. Using a certain modification of Sf he also obtained in [19] similar results for L_p, $1 \leqslant p < \infty$.

The purpose of this paper is to construct some multidimensional spline approximation methods based on the one-dimensional operator S. We construct these in two ways: by forming certain tensor products and by mimicking the methods in [6–8] for obtaining spline blended interpolation schemes.

We study questions of convergence for continuous and for L_p functions, rates of convergence for classes of smooth functions, and convergence of derivatives. In a later paper we shall discuss Jackson- and Bernstein-type theorems for these operators as well as analogs for the multidimensional case of the direct, inverse, and saturation theorems of Scherer.

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2. Schoenberg's Approximation Method

We review Schoenberg's approximation method for functions of one real variable (see [14, 16, 20]). Let m, n be integers with $m \geqslant 2$, $n \geqslant 1$, and let $\Delta = \{x_0, x_1, \ldots, x_{n+1}\}$, where $a = x_0 < x_1 < \cdots < x_{n+1} = b$. Let $x_{-m} = x_{1-m} = \cdots = x_0 = a$ and $b = x_{n+1} = \cdots = x_{n+m+1}$. For $i = -m, \ldots, n$ we define

$$\phi_i(x) = (x_{i+m+1} - x_i)\, M_m[x; x_i, \ldots, x_{i+m+1}], \tag{2.1}$$

where

$$M_m(x, y) = (y - x)_+^m$$

and $M[x; x_i, \ldots, x_{i+m+1}]$ denotes the $(m+1)^{st}$ divided difference of the function M_m with respect to the variable y over the points $x_i, \ldots, x_{i+m+1}$. The $\phi_i(x)$ are normalized B-splines of degree m with knots at the $\{x_i\}_0^{n+1}$.

For $j = -m, \ldots, n$ set

$$\xi_j = \frac{x_{j+1} + \cdots + x_{j+m}}{m}, \qquad \zeta_j = \frac{x_j + \cdots + x_{j+m}}{m+1}, \tag{2.2}$$

$$\xi_j^{(2)} = \frac{x_{j+1}x_{j+2} + \cdots + x_{j+m-1}x_{j+m}}{\binom{m}{2}}. \tag{2.3}$$

It is shown in [9, 14, 16] that for $a \leqslant x \leqslant b$

$$\sum_{-m}^{n} \phi_j(x) = 1, \tag{2.4}$$

$$\sum_{-m}^{n} \xi_j \phi_j(x) = x, \tag{2.5}$$

$$\sum_{-m}^{n} \xi_j^{(2)} \phi_j(x) = x^2, \tag{2.6}$$

$$\int_a^b \phi_j(x)\, dx = \frac{(x_{j+m+1} - x_j)}{m+1}, \qquad j = -m, \ldots, n, \tag{2.7}$$

and

$$\int_a^b |\phi_j(x)|^p\, dx \leqslant \frac{(x_{j+m+1} - x_j)}{m+1}, \qquad j = -m, \ldots, n, \tag{2.8}$$

whenever $1 \leqslant p < \infty$.

From the definition (2.2) it is clear that $a = \xi_{-m} < \cdots < \xi_n = b$. Given f defined on $[a, b]$, Schoenberg's approximation method is

$$S_{m,\Delta} f(x) = \sum_{-m}^{n} f(\xi_j)\, \phi_j(x). \tag{2.9}$$

We summarize some of the properties we need (see [14–16, 20]).

LEMMA 2.1. *Let $S_{m,\Delta}$ be defined as in* (2.9). *Then*

$$S_{m,\Delta} u_0(x) = u_0(x), \qquad S_{m,\Delta} u_1(x) = u_1(x), \tag{2.10}$$

where $u_0(t) \equiv 1$, $u_1(t) \equiv t$;

$$S_{m,\Delta} f(x) \geqslant 0 \qquad \textit{if} \quad f(t) \geqslant 0 \quad \textit{on } [a, b]; \tag{2.11}$$

$$0 \leqslant S_{m,\Delta} u_2(x) - u_2(x) \leqslant h^2(m, \Delta), \tag{2.12}$$

where $u_2(t) \equiv t^2$,

$$h(m, \Delta) = \min \left\{ \frac{b - a}{\sqrt{2m}}, ((m + 1)/12)^{1/2} \bar{\Delta} \right\}, \tag{2.13}$$

and $\bar{\Delta} = \max_{0 \leqslant j \leqslant n} (x_{j+1} - x_j)$;

$$\| f - S_{m,\Delta} f \|_\infty \to 0 \qquad \textit{iff} \quad \bar{\Delta}/m \to 0; \tag{2.14}$$

$$\| f - S_{m,\Delta} f \|_\infty \leqslant (1 + ((m + 1)/12)^{1/2})\, \omega(f; \bar{\Delta}), \tag{2.15}$$

and

$$\| f - S_{m,\Delta} f \|_\infty \leqslant \left(1 + \frac{b - a}{2^{1/2}}\right) \omega\left(f; \frac{1}{m^{1/2}}\right), \tag{2.16}$$

where $\omega(f; t)$ is the modulus of continuity of f;

$$\textit{if} \quad f \in C^1[a, b], \quad \bar{\Delta}/m \to 0, \qquad \textit{then} \quad \| (S_{m,\Delta} f - f)' \|_\infty \to 0. \tag{2.17}$$

We also have the following additional (new) properties.

LEMMA 2.2. *Let $S_{m,\Delta}$ be defined as in* (2.9). *Then if $f \in C^1[a, b]$,*

$$\| S_{m,\Delta} f - f \|_\infty \leqslant 2 \left(\frac{m + 1}{12}\right)^{1/2} \left(1 + \left(\frac{m + 1}{12}\right)^{1/2}\right) \bar{\Delta} \omega(f'; \bar{\Delta}) \tag{2.18}$$

and

$$\| S_{m,\Delta} f - f \|_\infty \leqslant 2 \left(\frac{b - a}{(2m)^{1/2}}\right)\left(1 + \frac{b - a}{2^{1/2}}\right) \omega\left(f'; \frac{1}{m^{1/2}}\right). \tag{2.19}$$

Moreover, if $f \in C^2[a, b]$, *then*

$$\| S_{m,\Delta} f - f \|_\infty \leqslant 2 \| f'' \|_\infty \left(\frac{m+1}{12} \right) \bar{\Delta}^2. \tag{2.20}$$

and

$$\| S_{m,\Delta} f - f \|_\infty \leqslant (b-a)^2 \| f'' \|_\infty / m. \tag{2.21}$$

Proof. It is shown in [3] (see also [5, 17]) that if L_m is a sequence of positive linear operators which reproduce linear functions, and if $f \in C^1[a, b]$, then

$$\| L_m f - f \|_\infty \leqslant 2\mu_m \omega(f'; \mu_m),$$

where

$$\mu_m = \| L_m(t-x)^2 \|_\infty^{1/2};$$

and that if $f \in C^2[a, b]$, then

$$\| L_m f - f \|_\infty \leqslant 2 \| f'' \|_\infty \mu_m^2.$$

For $L_m = S_{m,\Delta}$, we have by (2.12) that $\mu_m \leqslant h(m, \Delta)$, where h is defined in (2.13), and the results follow.

For approximation of functions $f \in L_p[a, b]$ it is useful to introduce the operator

$$\tilde{S}_{m,\Delta} f(x) = \sum_{-m}^{n} \phi_j(x) \int_{\zeta_j}^{\zeta_{j+1}} \frac{f(u)\, du}{(\zeta_{j+1} - \zeta_j)} \tag{2.22}$$

(see [19]).

3. Multivariate Approximation Methods

In this section we define some multivariate spline approximation methods based on the operators $S_{m,\Delta}$ and $\tilde{S}_{m,\Delta}$ defined in Section 2. It is convenient to consider only the case of two variables. We construct methods which lead to tensor product splines and certain spline blended functions (for the use of tensor product splines in multivariate interpolation see [1, 4]; spline blended interpolating functions were considered by Gordon [6–8]).

Let $I' = [a, b]$, $I'' = [c, d]$, and $R = I' \times I''$. Let $m_1, m_2 \geqslant 2$ be integers, and suppose $\Delta_1 = \{x_i\}_0^{n_1+1}$, $\Delta_2 = \{y_i\}_0^{n_2+1}$ are partitions of I' and I'' satisfying

$$\begin{gathered} a = x_0 < x_1 < \cdots < x_{n_1+1} = b \\ c = y_0 < y_1 < \cdots < y_{n_2+1} = d. \end{gathered} \tag{3.1}$$

We define $\{\xi_i\}_{-m_1}^{n_1}$ and $\{\xi_i^{(2)}\}_{-m_1}^{n_1}$ as in (2.2–3) and the functions $\{\phi_i(x)\}_{-m_1}^{n_1}$ as in (2.1). We denote by $\mathscr{S}_{m_1,\Delta_1}$ the space spanned by the ϕ_i. Let $\{\eta_j\}_{-m_2}^{n_2}$, $\{\eta_j^{(2)}\}_{-m_2}^{n_2}$, $\{\psi_j(y)\}_{-m_2}^{n_2}$, and $\mathscr{S}_{m_2,\Delta_2}$ be defined analogously for the partition Δ_2. Let $\{\gamma_j\}_{-m_2}^{n_2}$ be the analog of the ζ's for Δ_2.

We consider the following approximation methods for functions f defined on R:

$$S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f = \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} f(\xi_i, \eta_j)\, \phi_i(x)\, \psi_j(y), \tag{3.2}$$

$$\tilde{S}_{m_1,\Delta_1} \otimes \tilde{S}_{m_2,\Delta_2} f = \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} \frac{\phi_i(x)\, \psi_j(y) \int_{\zeta_i}^{\zeta_{i+1}} \int_{\gamma_j}^{\gamma_{j+1}} f(u, v)\, du\, dv}{(\zeta_{i+1} - \zeta_i)(\gamma_{j+1} - \gamma_j)}, \tag{3.3}$$

$$S_{m_1,\Delta_1} \oplus S_{m_2,\Delta_2} f = \sum_{-m_1}^{n_1} f(\xi_i, y)\, \phi_i(x) + \sum_{-m_2}^{n_2} f(x, \eta_j)\, \psi_j(y) - S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f, \tag{3.4}$$

and

$$\tilde{S}_{m_1,\Delta_1} \oplus \tilde{S}_{m_2,\Delta_2} f = \sum_{-m_1}^{n_1} \phi_i(x) \int_{\zeta_i}^{\zeta_{i+1}} \frac{f(u, y)\, du}{(\zeta_{i+1} - \zeta_i)} + \sum_{-m_2}^{n_2} \psi_j(y) \int_{\gamma_j}^{\gamma_{j+1}} \frac{f(x, v)\, dv}{(\gamma_{j+1} - \gamma_j)}$$
$$- \tilde{S}_{m_1,\Delta_1} \otimes \tilde{S}_{m_2,\Delta_2} f. \tag{3.5}$$

Whenever there is no possibility of ambiguity, we shall write subscripts 1 and 2 in place of m_1, Δ_1 and m_2, Δ_2 in (3.2–5); e.g., we write $S_1 \otimes S_2 = S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2}$.

The operators (3.2–3) map $C[R]$ into the subspace of tensor product splines

$$\mathscr{S}_{m_1,\Delta_1} \otimes \mathscr{S}_{m_2,\Delta_2} = \{f \in C^{(m_1-1,m_2-1)}[R]\text{: in each rectangle } [x_i, x_{i+1}]x$$
$$[y_j, y_{j+1}],\ f^{(m_1+1,0)} = f^{(0,m_2+1)} = 0,$$
$$i = 0, 1,..., n_1, j = 0, 1,..., n_2\}.$$

The operators (3.4–5) map $C[R]$ into the subspace of spline blended functions

$$\mathscr{S}_{m_1,\Delta_1} \oplus \mathscr{S}_{m_2,\Delta_2} = \operatorname{span}\{(\mathscr{S}_{m_1,\Delta_1} \otimes C[I'']) \cup (\mathscr{S}_{m_2,\Delta_2} \otimes C[I'])\}.$$

We emphasize that none of the operators (3.2–5) are projections.

Finally, we note

$$S_1 \otimes S_2 f = \sum_{-m_1}^{n_1} S_2[f(\xi_i, \cdot)](y)\, \phi_i(x) = \sum_{-m_2}^{n_2} S_1[f(\cdot, \eta_j)](x)\, \psi_j(y). \tag{3.6}$$

THEOREM 3.1. *The operators defined in* (3.2–5) *are linear. Moreover,*

$$S_1 \otimes S_2 g \equiv g \quad \text{for all} \quad g(x, y) = \alpha + \beta x + \gamma y + \delta xy \tag{3.7}$$

$$S_1 \oplus S_2 g \equiv g \quad \text{for all } g(x, y) = \alpha + x h_2(y) + y h_1(x), \qquad h_1, h_2 \text{ arbitrary} \tag{3.8}$$

the operators (3.2), (3.3) *are positive.* (3.9)

4. APPROXIMATION OF CONTINUOUS FUNCTIONS

In this section we discuss the behavior of the operators (3.2), (3.4) for functions $f \in C[R]$ as $m_1, m_2, \Delta_1, \Delta_2$ range over a sequence of values (usually the degrees m_1 or $m_2 \to \infty$ and/or the partitions Δ_1, Δ_2 are refined). We shall obtain rates of convergence for smooth functions as well as basic convergence results.

We discuss first the convergence question for the tensor spline method (3.2). We base our results on the fact that (by the triangle inequality)

$$\begin{aligned} \| S_1 \otimes S_2 f - f \| &= \left\| \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} [f(\xi_i, \eta_j) - f(x, y)]\, \phi_i(x)\, \psi_j(y) \right\| \\ &\leqslant \left\| \sum_{-m_2}^{n_2} \sum_{-m_1}^{n_1} [f(\xi_i, \eta_j) - f(x, \eta_j)]\, \phi_i(x)\, \psi_j(y) \right\| \\ &\quad + \left\| \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} [f(x, \eta_j) - f(x, y)]\, \psi_j(y)\, \phi_i(x) \right\| \\ &\leqslant \| S_1 f(\cdot, y)(x) - f(x, y) \| + \| S_2 f(x, \cdot)(y) - f(x, y) \|. \end{aligned} \tag{4.1}$$

(See also Schultz [21]).

We need the following multivariate version of a theorem of Bohman [2].

LEMMA 4.1. *Suppose* L_{n_1, n_2}, $n_1, n_2 = 1, 2, \ldots$ *is a sequence of operators of the form*

$$L_{n_1, n_2} f(x, y) = \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} f(\xi_{i, n_1}, \eta_{j, n_2})\, \Phi_{i, j, n_1, n_2}(x, y), \tag{4.2}$$

where $\Phi_{i,j,n_1,n_2}(x, y)$ *are any functions defined on* R *and*

$$a = \xi_{-m_1,n_1} < \cdots < \xi_{n_1,n_1} = b$$

$$c = \eta_{-m_2,n_2} < \cdots < \eta_{n_2,n_2} = d.$$

Then

$$\| L_{n_1,n_2} f - f \|_\infty \to 0 \quad \textit{for every} \quad f \in C[R] \quad \textit{as} \quad n_1, n_2 \to \infty \tag{4.3}$$

implies

$$\begin{cases} \max_i | \xi_{i+1,n_1} - \xi_{i,n_1} | \to 0 \\ \max_j | \eta_{j+1,n_2} - \eta_{j,n_2} | \to 0. \end{cases} \tag{4.4}$$

Our main convergence result for tensor splines is

THEOREM 4.2. *Suppose* m_1, m_2, Δ_1, Δ_2 *range over a sequence of values. Then*

$$\| S_1 \otimes S_2 f - f \|_\infty \to 0 \quad \textit{for all} \quad f \in C[R] \tag{4.5}$$

iff

$$\bar{\Delta}_1 / m_1 \to 0 \quad \textit{and} \quad \bar{\Delta}_2 / m_2 \to 0.$$

Proof. The sufficiency follows from (4.1) and (2.14). The necessity is a consequence of Lemma 4.1 and the fact that

$$\frac{\bar{\Delta}_1}{m_1} = \max_i \frac{(x_{i+1} - x_i)}{m_1} \leqslant \max_i \frac{(x_{i+m_1} - x_i)}{m_1} = \max_i (\xi_i - \xi_{i-1}),$$

and

$$\frac{\bar{\Delta}_2}{m_2} \leqslant \max_j (\eta_j - \eta_{j-1}).$$

For the blending spline functions we have

THEOREM 4.3. *Suppose* m_1, m_2, Δ_1, Δ_2 *range over a sequence of values such that* $\bar{\Delta}_1/m_1 \to 0$ *or* $\bar{\Delta}_2/m_2 \to 0$. *Then*

$$\| S_1 \oplus S_2 f - f \|_\infty \to 0 \quad \textit{for all} \quad f \in C[R]. \tag{4.6}$$

Proof. We have

$$\begin{aligned} (S_1 \oplus S_2 f - f)(x, y) &= \sum_{-m_1}^{n_1} \phi_i(x) [f(\xi_i, y) - f(x, y)] \\ &\quad + \sum_{-m_2}^{n_2} \psi_j(y) \left[f(x, \eta_j) - \sum_{-m_1}^{n_1} f(\xi_i, \eta_j)\, \phi_i(x) \right] \\ &= R_1(x, y) - (S_2 R_1(x, \cdot))(y), \end{aligned} \tag{4.7}$$

where

$$R_1(x, y) = \sum_{-m_1}^{n_1} \phi_i(x)[f(\xi_i, y) - f(x, y)] = (S_1 f(\cdot, y))(x) - f(x, y).$$

Now

$$\max_{(x,y)\in R} | R_1(x, y) - (S_2 R_1(x, \cdot))(y)| = \max_y | R_1(x^*, y) - S_2 R_1(x^*, \cdot)(y)|$$

for some $x^* \in [a, b]$. Applying (2.14) shows that $\bar{\Delta}_2/m_2 \to 0$ is sufficient for (4.6). The sufficiently of $\bar{\Delta}_1/m_1$ follows in a similar way.

More precise estimates of the rates of convergence of the operators (3.2) and (3.4) can be obtained in terms of the smoothness of f. For $f \in C[R]$ we define (see, e.g., [13])

$$\omega(f; h_1, h_2) = \max_{\substack{|x_2 - x_1| \leqslant h_1 \\ |y_2 - y_1| \leqslant h_2 \\ (x_1, y_1), (x_2, y_2) \in R}} |f(x_2, y_2) - f(x_1, y_1)|. \tag{4.8}$$

THEOREM 4.4. *Let $f \in C[R]$. Then we have*

$$\| S_1 \otimes S_2 f - f \|_\infty \leqslant \left(1 + \left(\frac{m_1 + 1}{12}\right)^{1/2}\right) \omega(f; \bar{\Delta}_1, 0) + \left(1 + \left(\frac{m_2 + 1}{12}\right)^{1/2}\right) \omega(f; 0, \bar{\Delta}_2), \tag{4.9}$$

$$\| S_1 \otimes S_2 f - f \|_\infty \leqslant \left(1 + \frac{b - a}{2^{1/2}}\right) \omega\left(f; \frac{1}{m_1^{1/2}}, 0\right) + \left(1 + \frac{d - c}{2^{1/2}}\right) \omega\left(f; 0, \frac{1}{m_2^{1/2}}\right), \tag{4.10}$$

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant 2 \min \left[\left(1 + \left(\frac{m_1 + 1}{12}\right)^{1/2}\right) \omega(f; \bar{\Delta}_1, 0), \left(1 + \left(\frac{m_2 + 1}{12}\right)^{1/2}\right) \omega(f; 0, \bar{\Delta}_2)\right], \tag{4.11}$$

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant 2 \min \left[\left(1 + \frac{b - a}{2^{1/2}}\right) \omega\left(f; \frac{1}{m_1^{1/2}}, 0\right), \left(1 + \frac{d - c}{2^{1/2}}\right) \omega\left(f; 0, \frac{1}{m_2^{1/2}}\right)\right]. \tag{4.12}$$

Proof. Applying (2.15) to (4.1) yields (4.9) while (2.16) leads to (4.10). To prove (4.11), we use (4.7) and (2.15). This yields

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant \sup_x \left(1 + \left(\frac{m_2 + 1}{12}\right)^{1/2}\right) \omega(R_1(x, \cdot); \bar{\Delta}_2)$$

But

$$\sup_x \omega(R_1(x, \cdot); \bar{\Delta}_2) = \sup_x \sup_{|y_2 - y_1| \leqslant \bar{\Delta}_2} | R_1(x, y_2) - R_1(x, y_1)| \leqslant 2\omega(f; 0, \bar{\Delta}_2).$$

Similarly, (4.12) follows from (2.16).

Using (2.18–19) we may prove in a similar way

THEOREM 4.5. *Let $f^{(1,0)}, f^{(0,1)} \in C[R]$. Then*

$$\| S_1 \otimes S_2 f - f \| \leqslant 2 \left(\frac{m_1 + 1}{12}\right)^{1/2} \left(1 + \left(\frac{m_1 + 1}{12}\right)^{1/2}\right) \bar{\Delta}_1 \omega(f^{(1,0)}; \bar{\Delta}_1 , 0) + 2 \left(\frac{m_2 + 1}{12}\right)^{1/2} \left(1 + \left(\frac{m_2 + 1}{12}\right)^{1/2}\right) \bar{\Delta}_2 \omega(f^{(0,1)}; 0, \bar{\Delta}_2) \tag{4.13}$$

$$\| S_1 \otimes S_2 f - f \|_\infty \leqslant 2 \left(\frac{b - a}{(2m_1)^{1/2}}\right) \left(1 + \frac{b - a}{2^{1/2}}\right) \omega \left(f^{(1,0)}; \frac{1}{m_1^{1/2}}, 0\right) + 2 \left(\frac{d - c}{(2m_2)^{1/2}}\right) \left(1 + \frac{d - c}{2^{1/2}}\right) \omega \left(f^{(0,1)}; 0, \frac{1}{m_2^{1/2}}\right), \tag{4.14}$$

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant 4 \min \left[\left(\frac{m_1 + 1}{12}\right)^{1/2} \left(1 + \left(\frac{m_1 + 1}{12}\right)^{1/2}\right) \bar{\Delta}_1 \omega(f^{(1,0)}; \bar{\Delta}_1 , 0), \left(\frac{m_2 + 1}{12}\right)^{1/2} \left(1 + \left(\frac{m_2 + 1}{12}\right)^{1/2}\right) \bar{\Delta}_2 \omega(f^{(0,1)}; 0, \bar{\Delta}_2)\right], \tag{4.15}$$

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant 4 \min \left[\left(\frac{b - a}{(2m_1)^{1/2}}\right) \left(1 + \frac{b - a}{2^{1/2}}\right) \omega \left(f^{(1,0)}; \frac{1}{m_1^{1/2}}, 0\right), \left(\frac{d - c}{(2m_2)^{1/2}}\right) \left(1 + \frac{d - c}{2^{1/2}}\right) \omega \left(f^{(0,1)}; 0, \frac{1}{m_2^{1/2}}\right)\right]. \tag{4.16}$$

Similarly, (2.20–21) leads to

THEOREM 4.6. *Let $f^{(2,0)}, f^{(0,2)} \in C[R]$. Then*

$$\| S_1 \otimes S_2 f - f \|_\infty \leqslant 2 \| f^{(2,0)} \|_\infty \left(\frac{m_1 + 1}{12}\right) \bar{\Delta}_1^2 + 2 \| f^{(0,2)} \|_\infty \left(\frac{m_2 + 2}{12}\right) \bar{\Delta}_2^2, \tag{4.17}$$

$$\| S_1 \otimes S_2 f - f \|_\infty \leqslant (b - a)^2 \| f^{(2,0)} \|_\infty / m_1 + (d - c)^2 \| f^{(0,2)} \|_\infty / m_2 , \tag{4.18}$$

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant 4 \min \left(\| f^{(2,0)} \|_\infty \left(\frac{m_1 + 1}{12} \right) \bar{\Delta}_1^2, \| f^{(0,2)} \|_\infty \left(\frac{m_2 + 1}{12} \right) \bar{\Delta}_2^2 \right), \tag{4.19}$$

$$\| S_1 \oplus S_2 f - f \|_\infty \leqslant 2 \min[(b - a)^2 \| f^{(2,0)} \|_\infty / m_1, (d - c)^2 \| f^{(0,2)} \|_\infty / m_2]. \tag{4.20}$$

In the following theorem we obtain an estimate for $S_1 \oplus S_2 f - f$ in terms of the measure of smoothness

$$\gamma(f; h_1, h_2) = \max_{\substack{|x_2 - x_1| \leqslant h_1 \\ |y_2 - y_1| \leqslant h_2 \\ (x_1, y_1), (x_2, y_2) \in R}} | f(x_2, y_2) + f(x_1, y_1) - f(x_2, y_1) - f(x_1, y_2) |, \tag{4.21}$$

defined for $f \in C[R]$, (cf. (4.8)).

THEOREM 4.7. *Let $f \in C[R]$. Then*

$$\| S_1 \oplus S_2 f - f \| \leqslant \left(1 + \left(\frac{m_1 + 1}{12} \right)^{1/2} \right) \left(1 + \left(\frac{m_2 + 1}{12} \right)^{1/2} \right) \gamma(f; \bar{\Delta}_1, \bar{\Delta}_2) \tag{4.22}$$

$$\| S_1 \oplus S_2 f - f \| \leqslant \left(1 + \frac{b - a}{2^{1/2}} \right) \left(1 + \frac{d - c}{2^{1/2}} \right) \gamma \left(f; \frac{1}{m_1^{1/2}}, \frac{1}{m_2^{1/2}} \right). \tag{4.23}$$

Proof. We have

$$\begin{aligned} \rho &= \| S_1 \oplus S_2 f - f \|_\infty \\ &= \left\| \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} [f(\xi_i, y) + f(x, \eta_j) - f(\xi_i, \eta_j) - f(x, y)] \varphi_i(x) \psi_j(y) \right\| \\ &\leqslant \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} \gamma(f, | x - \xi_i | \delta_1 \delta_1^{-1}, | y - \eta_j | \delta_2 \delta_2^{-1}) \varphi_i(x) \psi_j(y), \end{aligned}$$

where $\delta_1, \delta_2 > 0$ are any constants. By the triangle inequality,

$$\begin{aligned} \rho &\leqslant \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} \gamma(f; \delta_1, \delta_2)(1 + \delta_1^{-1} | x - \xi_i |)(1 + \delta_2^{-1} | y - \eta_j |) \varphi_i(x) \psi_j(y) \\ &= \gamma(f; \delta_1, \delta_2)\Big(1 + \delta_1^{-1} \sum_{-m_1}^{n_1} | x - \xi_i | \varphi_i(x)\Big)\Big(1 + \delta_2^{-1} \sum_{-m_2}^{n_2} | y - \eta_j | \psi_j(y)\Big). \end{aligned}$$

But the Hölder inequality implies $\sum_{-m_1}^{n_1} | x - \xi_i | \varphi_i(x) \leqslant h(m_1, \Delta_1)$ (cf. [15]). A similar estimate holds for the other sum. Choosing $\delta_1 = \bar{\Delta}_1$, $\delta_2 = \bar{\Delta}_2$ leads to (4.22), while $\delta_1 = 1/m_1^{1/2}$, $\delta_2 = 1/m_2^{1/2}$ results in (4.23).

5. Approximation of L_p Functions

In this section we discuss the behavior of the operators (3.3), (3.5) for functions $f \in L_p[R]$, $1 \leqslant p < \infty$, as the degrees m_1, m_2 and the partitions Δ_1, Δ_2 take on a sequence of values.

We discuss first the tensor spline operator (3.3).

THEOREM 5.1. *Let $f \in L_p[R]$. Suppose m_1, m_2, Δ_1, Δ_2 range over a sequence of values such that $\bar{\Delta}_1/m_1 \to 0$, $\bar{\Delta}_2/m_2 \to 0$. Then*

$$\| \tilde{S}_1 \otimes \tilde{S}_2 f - f \|_p \to 0. \tag{5.1}$$

Proof. We use the Banach-Steinhaus theorem and the fact that $C[R]$ is dense in $L_p[R]$. First we prove that $\tilde{S}_1 \otimes \tilde{S}_2$ are uniformly bounded (w.r.t. $m_1, m_2, \Delta_1, \Delta_2$) as operators from $L_p[R]$ into itself. Since $\tilde{S}_1 \otimes \tilde{S}_2 f = \tilde{S}_1 \tilde{S}_2 f$ if suffices to show that the operators

$$\tilde{S}_1 f(x, y) = \sum_{-m_1}^{n_1} \phi_i(x) \frac{\int_{\zeta_i}^{\zeta_{i+1}} f(u, y)\, du}{(\zeta_{i+1} - \zeta_i)} \tag{5.2}$$

and

$$\tilde{S}_2 f(x, y) = \sum_{-m_2}^{n_2} \psi_j(y) \frac{\int_{\gamma_j}^{\gamma_{j+1}} f(x, v)\, dv}{(\gamma_{j+1} - \gamma_j)} \tag{5.3}$$

are uniformly bounded from $L_p[R]$ into itself. We concentrate on (5.2).

Let

$$K(x, u) = \frac{\phi_i(x)}{\zeta_{i+1} - \zeta_i}, \qquad \zeta_i \leqslant u \leqslant \zeta_{i+1}, \qquad i = -m_1, \ldots, n_1.$$

It is easily seen that

$$\tilde{S}_1 f(x, y) = \int_a^b K(x, u) f(u, y)\, du.$$

Applying the Hölder inequality yields

$$\| \tilde{S}_1 f \|_p \leqslant \left\| \left[\int_a^b K(x, u) \mid f(u, y) |^p\, du \right]^{1/p} \left[\int_a^b K(x, u)\, du \right]^{1/q} \right\|_p.$$

But

$$\sup_x \int_a^b K(x, u)\, du \leqslant \sup_x \sum_{-m_1}^{n_1} \frac{\phi_i(x)}{\zeta_{i+1} - \zeta_i} \int_{\zeta_i}^{\zeta_{i+1}} du \leqslant 1,$$

while

$$\int_a^b \int_c^d \int_a^b K(x, u) \, |f(u, y)|^p \, du \, dx \, dy \leqslant \sup_u \int_a^b K(x, u) \, dx \, \|f\|_p^p$$

and

$$\sup_u \int_a^b K(x, u) \, dx = \sup_i \int_a^b \frac{\phi_i(x)}{\zeta_{i+1} - \zeta_i} \, dx \leqslant \sup_i \frac{(m_1 + 1)(x_{i+m_1+1} - x_i)}{(m_1 + 1)(x_{i+m_1+1} - x_i)} = 1.$$

Similarly, we have $\| \tilde{S}_2 f \|_p \leqslant \| f \|_p$.

To complete the proof of Theorem 5.1, we show that (5.1) holds for $f \in C[R]$. We have

$$\begin{aligned}
&\| \tilde{S}_1 \otimes \tilde{S}_2 f - f \|_p \\
&\quad \leqslant \| f - S_1 \otimes S_2 f \|_p + \Big\| \sum_{-m_1}^{n_1} \sum_{-m_2}^{n_2} \frac{\phi_i(x) \, \psi_j(y)}{(\zeta_{i+1} - \zeta_i)(\gamma_{j+1} - \gamma_j)} \\
&\qquad \times \int_{\zeta_i}^{\zeta_{i+1}} \int_{\gamma_j}^{\gamma_{j+1}} [f(\zeta_i, \gamma_j) - f(u, v)] \, du \, dv \Big\|_p \\
&\quad \leqslant (b - a)^{1/p} \| f - S_1 \otimes S_2 f \|_\infty + \omega(f; \max_i (\zeta_{i+1} - \zeta_i), \max_j (\gamma_{j+1} - \gamma_j)).
\end{aligned}$$

Now $(\zeta_{i+1} - \zeta_i) \leqslant (\xi_{i+1} - \xi_{i-1})$ which goes to zero by Lemma 4.1.

By Theorem 4.2, the properties of modulus of continuity, and the hypotheses, all of these terms go to zero, and the theorem is proved.

THEOREM 5.2. *Let $f \in L_p[R]$. Suppose m_1, m_2, Δ_1, Δ_2 range over a sequence of values such that $\Delta_1 m_1 \to 0$ or $\Delta_2 m_2 \to 0$. Then*

$$\| \tilde{S}_1 \oplus \tilde{S}_2 f - f \|_p \to 0. \tag{5.4}$$

Proof. Clearly $\tilde{S}_1 \oplus \tilde{S}_2 = \tilde{S}_1 + \tilde{S}_2 - \tilde{S}_1 \otimes \tilde{S}_2$ are uniformly bounded operators from $L^p[R]$, (see the proof of Theorem 5.1). To complete the proof, we show (5.4) holds for $f \in C[R]$. In analogy with (4.7) we have

$$(\tilde{S}_1 \oplus \tilde{S}_2 f - f)(x, y) = \tilde{R}_1(x, y) - (\tilde{S}_2 \tilde{R}_1(x, \cdot)(y) \tag{5.5}$$

where

$$\tilde{R}_1(x, y) = (\tilde{S}_1 f(\cdot, y))(x) - f(x, y). \tag{5.6}$$

Then

$$\begin{aligned}
\| \tilde{S}_1 \oplus \tilde{S}_2 f - f \|_p^p &\leqslant \| \tilde{R}_1(x, y) - (\tilde{S}_2 \tilde{R}_1(x, \cdot))(y) \|_p^p, \\
&\leqslant (b - a) \sup_x \int_c^d | \tilde{R}_1(x, y) - \tilde{S}_2 \tilde{R}_1(x, \cdot)(y) |^p \, dy \\
&= \int_c^d | \tilde{R}_1(x^*, y) - \tilde{S}_2 \tilde{R}_1(x^*, \cdot)(y) |^p \, dy (b - a)
\end{aligned}$$

for some $x^* \in [a, b]$. The theorem follows from the fact that for any $g \in L_p[I'']$, $\| \tilde{S}_2 g - g \|_p^p \to 0$ provided $\bar{\Delta}_2/m_2 \to 0$. (The proof of this proceeds exactly as in the multidimensional Theorem 5.1.)

6. Convergence of Derivatives

To discuss the derivatives of the splines (3.2–3) we need further notation. Let $\Delta_1 = \{x_0, x_1, ..., x_{n_1+1}\}$ be a partition of $[a, b]$ satisfying

$$x_{-m_1} = \cdots = x_0 = a < x_1 < \cdots < x_{n_1} < b = x_{n_1+1} = \cdots = x_{n_1+m_1+1}.$$

Then we define for $i = 1 - m_1, ..., n_1$

$$\phi_i^-(x) = (x_{i+m_1} - x_i)\, M_{m_1-1}[x; x_i, ..., x_{i+m_1}], \tag{6.1}$$

$$\xi_i^- = \frac{(x_{i+1} + \cdots + x_{i+m_1-1})}{m_1 - 1}. \tag{6.2}$$

Similarly we define for $i = 2 - m_1, ..., n_1$

$$\phi_i^=(x) = (x_{i+m_1-1} - x_i)\, M_{m_1-2}[x; x_i, ..., x_{i+m_1-1}], \tag{6.3}$$

$$\xi_i^= = \frac{(x_{i+1} + \cdots + x_{i+m_1-2})}{m_1 - 2}. \tag{6.4}$$

We note that $\xi_{i-1} \leqslant \xi_i^- \leqslant \xi_i$ and $\xi_{i-1}^- \leqslant \xi_i^= \leqslant \xi_i^-$. The ϕ_i^- and $\phi_i^=$ are B-splines of degree $m_1 - 1$ and $m_1 - 2$, respectively. Corresponding to a partition Δ_2 of $[c, d]$ we define $\psi_j^-(y)$, $\psi_j^=(y)$, η_j^- and $\eta_j^=$ analogously. Then for example, S_{m_1-1,Δ_1} is the operator

$$S_{m_1-1,\Delta_1} g(x) = \sum_{1-m_1}^{n_1} \phi_i^-(x)\, g(\xi_i^-).$$

We discuss the case of the partial derivative with respect to x.

Lemma 6.1. *Let $f^{(1,0)} \in C[R]$. Then*

$$[S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f]^{(1,0)}(x, y) = \sum_{1-m_1}^{n_1} \sum_{-m_2}^{n_2} \frac{[f(\xi_i, \eta_j) - f(\xi_{i-1}, \eta_j)]}{(\xi_i - \xi_{i-1})}\, \phi_i^-(x)\, \psi_j(y). \tag{6.5}$$

Moreover, there exists a function $r(x, y) \in C[R]$ such that

$$\max_R | r(x, y)| \leqslant \omega(f^{(1,0)}; \max_i(\xi_{i+1} - \xi_i), 0) \tag{6.6}$$

and

$$[S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f]^{(1,0)}(x, y) = S_{m_1-1,\Delta_1} \otimes S_{m_2,\Delta_2}[f^{(1,0)} + r](x, y). \quad (6.7)$$

Proof. To obtain (6.5) we apply Lemma 1 of [14] to the expression (3.6). By the mean value theorem

$$\frac{f(\xi_i, \eta_j) - f(\xi_{i-1}, \eta_j)}{(\xi_i - \xi_{i-1})} = f^{(1,0)}(\tilde{\xi}_i, \eta_j),$$

where $\xi_{i-1} \leqslant \tilde{\xi}_i \leqslant \xi_i$. Thus (6.7) holds with

$$r(\xi_i^-, \eta_j) = f^{(1,0)}(\tilde{\xi}_i, \eta_j) - f^{(1,0)}(\xi_i^-, \eta_j). \quad (6.8)$$

Clearly we can choose $r(x, y) \in C[R]$ satisfying (6.8) such that (6.6) holds.

THEOREM 6.2. *Let $f^{(1,0)} \in C[R]$. Suppose m_1, m_2, Δ_1, Δ_2 range over a sequence of values such that $\bar{\Delta}_1/m_1 \to 0$ and $\bar{\Delta}_2/m_2 \to 0$. Then*

$$\| [S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f - f]^{(1,0)} \|_\infty \to 0. \quad (6.9)$$

Proof. By (6.7)

$$\| [S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f - f]^{(1,0)} \|_\infty = \| S_{m_1-1,\Delta_1} \otimes S_{m_2,\Delta_2} f^{(1,0)} - f^{(1,0)} \|_\infty$$
$$+ \| S_{m_1-1,\Delta_1} \otimes S_{m_2,\Delta_2} r \|_\infty . \quad (6.10)$$

The first term goes to zero by Theorem 4.2. For the second term we have

$$\| S_{m_1-1,\Delta_1} \otimes S_{m_2,\Delta_2} r \|_\infty \leqslant \| r \|_\infty \leqslant \omega(f^{(1,0)}; \max_i(\xi_{i+1} - \xi_i), 0). \quad (6.11)$$

Since the hypotheses of this theorem imply $S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f$ converges uniformly to f, Lemma 4.1 implies $\max(\xi_{i+1} - \xi_i) \to 0$, and the proof is complete.

We can also give a rate of convergence result for the derivative.

THEOREM 6.3. *Let $f^{(1,0)} \in C[R]$. Then*

$$\| [S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f - f]^{(1,0)} \|_\infty \leqslant \left(2 + \left(\frac{m_1 + 1}{12}\right)^{1/2}\right) \omega(f^{(1,0)}; \bar{\Delta}_1, 0)$$
$$+ \left(1 + \left(\frac{m_2 + 1}{12}\right)^{1/2}\right) \omega(f^{(1,0)}; 0, \bar{\Delta}_2). \quad (6.12)$$

Proof. We combine (6.10), (6.11) and Theorem 4.4, noticing that $\max_i (\xi_{i+1} - \xi_i) \leqslant \bar{\Delta}_1$.

The analogs of Theorem 6.2, 6.3 hold also for the (0, 1)-derivative if $f^{(0,1)} \in C[R]$. We now consider second derivatives.

LEMMA 6.4. *Let $f^{(2,0)} \in C[R]$. Then*

$$[S_1 \otimes S_2 f]^{(2,0)}(x, y) = \sum_{2-m_1}^{n_1} \sum_{-m_2}^{n_2} \nabla_\xi^2 f(\xi_i, \eta_j)\, \phi_i^=(x)\, \psi_j(y),$$

$$= \sum_{2-m_1}^{n_1} \sum_{-m_2}^{n_2} f^{(2,0)}(\tilde{\xi}_i, \eta_j) \frac{(\xi_i - \xi_{i-2})}{2(\xi_i^- - \xi_{i-1}^-)}\, \phi_i^=(x)\, \psi_j(y), \tag{6.13}$$

where

$$\nabla_\xi^2 f(\xi_i, \eta_j) = \frac{\dfrac{f(\xi_i, \eta_j) - f(\xi_{i-1}, \eta_j)}{(\xi_i - \xi_{i-1})} - \dfrac{f(\xi_{i-1}, \eta_j) - f(\xi_{i-2}, \eta_j)}{(\xi_{i-1} - \xi_{i-2})}}{(\xi_i^- - \xi_{i-1}^-)},$$

and $\xi_{i-2} < \tilde{\xi}_i < \xi_i$.

Proof. We apply Lemma 2 of [14, p. 35] to (3.6).

To prove a convergence result, we need a multivariate version of another result of Bohman [2], which we state without proof.

LEMMA 6.5. *Suppose L_{n_1,n_2}, $n_1, n_2 = 1, 2,...$ is a sequence of operators of the form* (4.6) *with $\Phi_{i,j,n_1,n_2}(x, y) \geqslant 0$, and suppose* (4.3) *holds. Then given any $\delta_1, \delta_2 > 0$*

$$\sum_{|\xi_{i,n_1} - x| < \delta_1} \sum_{|\eta_{j,n_2} - y| < \delta_2} \Phi_{i,j,n_1,n_2}(x, y) \to 1 \tag{6.14}$$

and

$$\sum_{|\xi_{i,n_1} - x| \geqslant \delta_1} \sum_{|\eta_{j,n_2} - y| \geqslant \delta_2} \Phi_{i,j,n_1,n_2}(x, y) \to 0 \tag{6.15}$$

uniformly in $(x, y) \in R$ as $n_1, n_2 \to \infty$.

THEOREM 6.6. *Let $f^{(2,0)} \in C[R]$. Let m_1, m_2 be fixed and suppose*

$$\Delta_1 = \left\{ x_i = a + \frac{(b-a)i}{(n_1+1)} \right\}_{i=0}^{n_1+1} \quad \text{and} \quad \Delta_2 = \left\{ y_j = c + \frac{(d-c)j}{(n_2+1)} \right\}_{j=0}^{n_2+1}.$$

Suppose $n_1, n_2 \to \infty$. Then for any compact subset K contained in the interior of R,

$$\| [S_{m_1,\Delta_1} \otimes S_{m_2,\Delta_2} f - f]^{(2,0)} \|_K \to 0$$

where $\| \cdot \|_K$ denotes the uniform norm on K.

Proof. By Lemma 6.4

$$E_2(x, y) \triangleq [S_1 \otimes S_2 f - f]^{(2,0)}$$

$$= \sum_{2-m_1}^{n_1} \sum_{-m_2}^{n_2} \phi_i^{=}(x)\, \psi_j(y)[f^{(2,0)}(\tilde{\xi}_i, \eta_j)\, B_i - f^{(2,0)}(x, y)], \quad (6.16)$$

where $B_i = (\xi_i - \xi_{i-2})/2(\xi_i^- - \xi_{i-1}^-)$. Let $\tilde{R}$ be a rectangle such that $K \subset \tilde{R} \subset$ interior (R). Given $\epsilon > 0$ we show that there exists N_1, N_2 such that $| E_2(x, y)| < (m_1 \| f^{(2,0)} \|_\infty + 1)\epsilon$ for $(x, y) \in \tilde{R}$ if $n_i > N_i$, $i = 1, 2$.

We collect some facts. Since $n_1, n_2 \to \infty$ the splines $S_{m_1-2,\Delta_1} \otimes S_{m_2,\Delta_2} f$ converge for every $f \in C[R]$. Thus by Lemma 4.1, $\xi_i^= - \xi_{i-1}^= \to 0$ and $\eta_j - \eta_{j-1} \to 0$ as $n_1, n_2 \to \infty$. Moreover, by Lemma 6.5, given $\delta_1, \delta_2 > 0$ there exist N_1, N_2 such that $n_i > N_i$, $i = 1, 2$, implies

$$\sum_{|\xi_i^= - x| \geqslant \delta_1} \; \sum_{|\eta_j - y| \geqslant \delta_2} \phi_i^{=}(x)\, \psi_j(y) < \epsilon.$$

For $i = 1, 2,..., n_1 - m_1$, $B_i \equiv 1$. Also $B_i \leqslant m_1 - 1$.

We split the sum in (6.16) into two parts. First

$$\left| \sum_{|x - \xi_i^=| \geqslant \delta_1} \; \sum_{|y - \eta_j| \geqslant \delta_2} \right| \leqslant m_1 \| f^{(2,0)} \|_\infty \, \epsilon.$$

Also if n_1 is sufficiently large

$$\left| \sum_{|x - \xi_i^=| < \delta_1} \; \sum_{|y - \eta_j| < \delta_2} \right| \leqslant \omega\, (f^{(2,0)};\ \max |\, x - \tilde{\xi}_i \,|, 0).$$

Now $\max_i |\, x - \tilde{\xi}_i \,| \leqslant 2/n_1 + \delta_1$. Thus we may choose δ_1, δ_2 sufficiently small (and N_1, N_2 sufficiently large) so that

$$|\, E_2(x, y)| \leqslant (m_1 \| f^{(2,0)} \|_\infty + 1)\epsilon.$$

As in the one dimensional case (see [14, p. 36]), $S_1 \otimes S_2 f^{(2,0)}$ does not converge uniformly to $f^{(2,0)}$ on all of R. Similar results hold for the $(0, 2)$ and $(1, 1)$ derivatives. Finally, we state without proof parallel results for the operator $S_1 \oplus S_2$.

THEOREM 6.7. *Suppose $f^{(1,0)} \in C[R]$ and that m_1, m_2, Δ_1, Δ_2 range over a sequence of values such that $\bar{\Delta}_1/m_1 \to 0$ or $\bar{\Delta}_2/m_2 \to 0$. Then*

$$\|(S_1 \oplus S_2 f - f)^{(1,0)} \|_\infty \to 0.$$

THEOREM 6.8. *Theorem 6.6 holds for $S_1 \oplus S_2$ with either $n_1 \to \infty$ or $n_2 \to \infty$.*

7. Remarks

1. Schoenberg's approximation method (2.9) reduces to the mth degree Bernstein polynomial in case $n = 0$. Similarly, we can also define the operator (3.2) for $n_1 = n_2 = 0$ and it will coincide with the (m_1, m_2)-degree two-dimensional Bernstein polynomial (see [12, 21]). The operator (3.2) can also be defined for $m_1 = m_2 = 1$, but it reduces to the bilinear interpolating spline so we have excluded this case.

2. The $\phi_i(x)$ and $\psi_j(y)$ in the operators (3.2–5) can be replaced by Tchebycheffian B-splines as in [10, 11, 14]. Many of our results hold for the resulting T-spline operators.

3. Throughout this paper we have assumed that the points in the partitions Δ_1, Δ_2 were distinct. This can be replaced by the requirement that at most m_1 x's coalesce in Δ_1 or m_2 x's in Δ_2 without difficulty. As in [10, 11, 14] the splines ϕ_i and ψ_j then have multiple knots.

4. As in the one dimensional case, it may be desirable to specify the nodes $\{\xi_i\}_{-m_1}^{n_1}$ and $\{\eta_j\}_{-m_2}^{n_2}$ rather than the partitions Δ_1, Δ_2. The question of when specified nodes can be achieved by admissible collections of knots reduces to the one-dimensional case studied in [10, 11, 14].

5. For ease of exposition we have concentrated on the case of two dimensions. Analogous multivariate approximation methods can also be defined for an arbitrary dimension d, and our results extend easily. For $d > 2$ it is possible to construct a variety of other operators intermediate between the tensor product $S_1 \otimes S_2$ (which is an analog of the minimal approximation methods) and the d-dimensional version of the Boolean sum $S_1 \oplus S_2$ (which is an analog of the maximal approximation methods) [6–8].

6. Theorem 4.2 can also be based on a multidimensional version of Volkov's Theorem [22] and the fact that $\| S_1 \otimes S_2[t^2 + s^2](x, y) - x^2 - y^2 \| \leqslant h^2(m_1, \Delta_1) + h^2(m_2, \Delta_2)$.

7. Theorem 4.7 was suggested by the referee.

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